International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Quantum Observables in Classical Frameworks

E. G. Beltrametti[1] **and S. Bugajski**[2]

Received March 28, 1995

A procedure of classical extension of a theory is worked out on the basis of a natural generalization of the notion of observable, the states of the extended theory being the probability measures on the pure states of the original one. Such a classical extension applies to quantum theory, and the qualifying features of quantum observables are preserved in the extended model.

1. INTRODUCTION

We deal with a new kind of classical representation, or classical extension, of quantum theories which rests on a physically natural notion of observable that encompasses both the usual classical and quantum versions. This classical extension involves the fact that, starting from a quantum set of states S_Q, it is always possible to construct a new classical structure of states—namely a convex set where the nonpure states have a unique decomposition into pure ones—which can be mapped onto S_Q.

The roots of this study can be found in previous work of one of the present authors (Bugajski, 1993). The use of probability measures on the pure elements of S_Q to represent the states is not new: it was considered by Misra (1974) and Ghirardi *et al.* (1976), and sometimes appears in the literature, as for instance in a recent paper by Amann (1993). The usual quantum observables, when looked upon in this classical extension, appear as unsharp classical observables, and the qualifying quantum features are preserved in this extension.

[1] Department of Physics, University of Genoa, and Istituto Nazionale di Fisica Nucleare, Sezione di Genova, I-16146 Genoa, Italy.
[2] Institute of Physics, University of Silesia, PL 40 007 Katowice, Poland.

2. ON THE NOTION OF OBSERVABLE

The intuitive physical notion of observable consists in the specification of the possible outcomes and of their probability distribution for each state of the physical system. Let S be a convex set representing, in a given theoretical model, the set of all states of the physical system under attention; let Ξ be a measurable space (in which the observable will take values) and write $\mathcal{B}(\Xi)$ for the Boolean σ-algebra of the measurable subsets of Ξ. We shall then define an observable, in the given theoretical model, as an affine map of S into the convex set $M_1^+(\Xi)$ of the probability measures on Ξ. Typically we shall have for Ξ the real line $\mathbb{R}$, but also $\Xi = \mathbb{R}^n$ might occur when dealing, for instance, with joint observables. All the singletons of Ξ will be assumed to be measurable.

This definition of observable, though natural, is not of frequent use in the literature: one can find it applied in studies on probabilistic and statistical aspects of quantum mechanics (see, e.g., Holevo, 1982), in works on the phase-space representation of quantum theories (see, e.g., Ali and Prugovecki, 1977; Singer and Stulpe, 1992; Bugajski, 1993), or in studies on more general frameworks (Beltrametti and Bugajski, 1993; Cassinelli and Lahti, 1993). As we shall see, this definition of observable encompasses the usual versions used in classical or in quantum mechanics.

Let B: $S \to M_1^+(\Xi)$ be an observable; to any pair (α, X), $\alpha \in S$, $X \in \mathcal{B}(\Xi)$, it associates the real number $(B\alpha)(X) \in [0, 1]$, i.e., the value the measure $B\alpha$ takes at the set X. For fixed $X \in \mathcal{B}(\Xi)$ we get an affine function $E_{B,X}$ from S into $[0, 1]$, hence an *effect* according to the common terminology.

The effects form a poset under the pointwise ordering: if a_1, a_2 are effects, we say that $a_1 \leq a_2$ whenever $a_1(\alpha) \leq a_2(\alpha)$ for all $\alpha \in S$. We write 0_S for the least effect (the null function on S) and e_S for the greatest effect (the unit function on S); we denote $[0_S, e_S]$ the set of all effects on S, and notice that it is naturally endowed with a convex structure, for the convex combination of two effects is obviously an effect.

The observables can be identified with the effect-valued measures on $\mathcal{B}(\Xi)$: in fact, any observable determines an effect-valued measure on $\mathcal{B}(\Xi)$ and, conversely, any effect-valued measure on $\mathcal{B}(\Xi)$ clearly determines an affine map of S into $M_1^+(\Xi)$, hence an observable.

Let us now look at a number of familiar notions associated with the definition of observable.

(i) *Spectrum.* Intuitively, the spectrum of the observable B: $S \to M_1^+(\Xi)$, denoted SpB, is the smallest subset of Ξ that contains all possible values of B. More precisely, we need $\mathcal{B}(\Xi)$ to be generated in the standard way by a topology on Ξ (as in the case $\Xi = \mathbb{R}$) and SpB is defined as the smallest among the closed subsets of Ξ such that $E_{B,SpB} = e_S$.

(ii) *Expectation value and variance.* Assuming for Ξ some linear structure (take, e.g., $\Xi = \mathbb{R}$), let $\mu \in M_1^+(\Xi)$, and define as usual its expectation and variance by

$$Exp(\mu) := \int_\Xi \xi \, d\mu(\xi); \qquad Var(\mu) := \int_\Xi [\xi - Exp(\mu)]^2 \, d\mu(\xi) \tag{1}$$

provided the integrals exist (which is ensured if Ξ is bounded). If μ is the image of a state $\alpha \in S$ under the observable B, we call the quantities in (1) the expectation value (or mean value) and the variance of the observable B at the state α and we favor the notation $Exp(B, \alpha)$ and, respectively, $Var(B, \alpha)$.

(iii) *Eigenstates and eigenvalues.* An eigenstate of the observable B: $S \to M_1^+(\Xi)$ is a state $\alpha \in S$ which is mapped by B into a probability measure concentrated at some $\lambda \in \Xi$. We write δ_λ for such a probability measure and call it a Dirac measure. λ is said to be an eigenvalue of B.

(iv) *Sharpness.* It may happen that the effects associated to an observable B belong to the family $\partial[0_S, e_S]$ of the extremal elements of the convex set $[0_S, e_S]$. In this case the observable is said to be sharp.

(v) *Uncertainty relations.* We say that two observables B_1: $S \to M_1^+(\Xi_1)$ and B_2: $S \to M_1^+(\Xi_2)$ obey an uncertainty relation (or are complementary) if there is a positive h such that

$$Var(B_1, \alpha) \, Var(B_2, \alpha) \geq h \tag{2}$$

for all α's at which the two variances exist.

(vi) *Comeasurability.* Two observables B_1: $S \to M_1^+(\Xi_1)$ and B_2: $S \to M_1^+(\Xi_2)$ are called comeasurable if there exists a third observable B: $S \to M_1^+(\Xi_1 \times \Xi_2)$ such that $B_1 = \pi_1 \circ B$, $B_2 = \pi_2 \circ B$, where $\Xi_1 \times \Xi_2$ is the measurable-space product of Ξ_1, Ξ_2, while π_1 and π_2 are the marginal projections of $M_1^+(\Xi_1 \times \Xi_2)$ onto $M_1^+(\Xi_1)$ and $M_1^+(\Xi_2)$, respectively. We recall that a marginal projection, say π_1, of $M_1^+(\Xi_1 \times \Xi_2)$ is defined by $(\pi_1\mu)(X) := \mu(X \times \Xi_2)$ for any $\mu \in M_1^+(\Xi_1 \times \Xi_2)$ and $X \in \mathcal{B}(\Xi_1)$. The observable B above is called the joint observable of B_1 and B_2, while $B\alpha$, $\alpha \in S$, is traditionally called the joint probability distribution of B_1 and B_2 at α. Notice that the joint observable of B_1 and B_2 need not be unique (Beltrametti and Bugajski, 1994).

We shall now verify that the definitions of observables commonly used in the quantum and the classical cases are recovered by our definition.

The Quantum Case. S becomes the set S_Q of all density operators on a separable complex Hilbert space $\mathcal{H}$, and the effects on S_Q are known to be in one-to-one correspondence with the positive operators on $\mathcal{H}$ which have mean value at every state not bigger than one: explicitly, if $\mathcal{P}$ is such an operator and $D \in S_Q$, then the effect associated to $\mathcal{P}$ is the function $S_Q \to [0, 1]$ defined by $Tr(D\mathcal{P})$. We account for this fact by saying that the observables are the POV-measures.

On the other hand, in the standard formulation of quantum mechanics the observables are defined as the self-adjoint operators on $\mathcal{H}$, namely as the projection-valued measures (PV-measures) on $\mathcal{B}(\mathbb{R})$, and PV-measures are obviously a subclass of POV-measures: thus the usual "observables" of quantum mechanics are recovered as a particular case of the observables defined more generally as the affine functions from S_Q into $M_1^+(\mathbb{R})$. It is worth remarking that the effects associated to projection operators on $\mathcal{H}$ can be viewed (see, e.g., Davies, 1976, p. 19) as the extremal elements of the convex set $[0_{S_Q}, e_{S_Q}]$. Thus, the usual observables of quantum mechanics correspond to the sharp observables in the more general framework we are considering.

The Classical Case. S becomes the simplex $M_1^+(\Omega)$ of all probability measures on some measurable space Ω, the "phase space" of the physical system under discussion, whose elements can be thought of as the pure states (we assume the singletons of Ω to be measurable). The convex set $S = M_1^+(\Omega)$ meets the most essential property of classical (as opposed to quantum) physical systems: the unique decomposability of mixed states into pure states. According to our definition, the observables are now the affine mappings of $M_1^+(\Omega)$ into $M_1^+(\Xi)$.

On the other hand, in classical statistical mechanics the observables are commonly represented by measurable functions on Ω with values in Ξ. Let f: $\Omega \to \Xi$ be one of such functions and define B_f: $M_1^+(\Omega) \to M_1^+(\Xi)$ by

$$(B_f\nu)(X) := \nu(f^{-1}(X)) \tag{3}$$

where $\nu \in M_1^+(\Omega)$, $X \in \mathcal{B}(\Xi)$, and $f^{-1}(X)$ is the counterimage of X under f. It is easily seen that B_f is affine: hence it is an observable. We come to the conclusion that the usual observables of classical statistical mechanics are recovered as a particular case.

As we shall see, a special role is played by those effects on $M_1^+(\Omega)$ which come from measurable functions on Ω taking values in $[0, 1]$. Indeed, let g be one of these functions and attach to it the function a_g: $M_1^+(\Omega) \to [0, 1]$ defined by

$$a_g(\nu) = \int_\Omega g(\omega)\, d\nu(\omega) \tag{4}$$

a_g is affine, hence it is an effect that we shall call regular. Correspondingly, an observable B: $M_1^+(\Omega) \to M_1^+(\Xi)$ will be called regular whenever its effects $E_{B,X}$ are regular for every $X \in \mathcal{B}(\Xi)$. The usual observables of classical statistical mechanics belong to the family of regular observables (Beltrametti and Bugajski, 1994): they are the sharp ones or, equivalently, their effects have the form (4), where g is a characteristic function χ_Y for some $Y \in \mathcal{B}(\Xi)$.

With reference to the notion of comeasurability given above, it is worth noting the following property (Beltrametti and Bugajski, 1994): Any two regular observables B_1: $M_1^+(\Omega) \to M_1^+(\Xi_1)$, B_2: $M_1^+(\Omega) \to M_1^+(\Xi_2)$ are comeasurable.

3. MODEL EXTENSION

Given the description of a physical system in terms of a convex set of states S and of the observables on S as discussed in the previous section, we want now to introduce the notion of extension of this descriptive model, based on some new convex set, say $\tilde{S}$. Thus we say that the convex set $\tilde{S}$ provides an extension of the model based on S if there exists an affine surjective map R: $\tilde{S} \to S$. Such a map will be called the reduction map, for it reduces, so to speak, the $\tilde{S}$-based model to the S-based one.

As a familiar example one could think, in the framework of standard quantum mechanics, of a compound system and a subsystem of it: the description of the subsystem on the basis of the Hilbert space of the compound system provides an extension of the description based on Hilbert space of the subsystem and the partial trace provides the reduction map. More generally, the above definition of model extension captures the physical notion of "coarse graining": since R maps $\tilde{S}$ onto S, it determines a partition of $\tilde{S}$ into equivalence classes, in the sense that all elements of $\tilde{S}$ having the same image in S form an equivalence class, or a coarse graining.

Let us focus attention on the observables of the S-based and of the $\tilde{S}$-based models. Loosely speaking, since $\tilde{S}$ is "richer" than S, we expect to have "more" observables on $\tilde{S}$ than on S. Actually every observable on S has a representative on $\tilde{S}$ through the reduction map. In fact, if B: $S \to M_1^+(\Xi)$ is an observable, then the map composition $B \circ R$: $\tilde{S} \to S \to M_1^+(\Xi)$ is an observable on $\tilde{S}$, to be denoted $\tilde{B}$. The observables on $\tilde{S}$ which are representatives of the ones on S share a surprisingly wide array of properties of the original observables. We shall list below the main invariants between corresponding observables.

1. If $B\alpha$ is the probability measure on Ξ describing the statistical distribution of results of measurements of B at α, predicted by the S-based model, then $\tilde{B}\tilde{\alpha} = B\alpha$ for every $\tilde{\alpha}$ in the counterimage of α under R: thus, the extended model based on $\tilde{S}$ predicts the same statistical distribution of results of measurements of $\tilde{B}$ at $\tilde{\alpha}$. In short, $BS = \tilde{B}\tilde{S}$ for every B: $S \to M_1^+(\Xi)$. In particular, two corresponding observables B, $\tilde{B}$ have the same spectrum, and the same eigenvalues [see (i), (iii) of Section 2]: intuitively, this means that

both the original observable B on S and its representative $\tilde{B}$ on $\tilde{S}$ have the same set of possible outcomes. Moreover, if the observable B on S has expectation $Exp(B, \alpha)$ and variance $Var(B, \alpha)$ at the state $\alpha \in S$ [see (ii) of the previous section], then

$$Exp(\tilde{B}, \tilde{\alpha}) = Exp(B, \alpha), \qquad Var(\tilde{B}, \tilde{\alpha}) = Var(B, \alpha) \tag{5}$$

for every $\tilde{\alpha}$ in the counterimage of α under R.

2. Let B_1, B_2 be two observables on S that satisfy the uncertainty relation

$$Var(B_1, \alpha)\ Var(B_2, \alpha) \geq h$$

for all $\alpha \in S$ at which the two variances exist. In view of equation (5) and taking note that R maps $\tilde{S}$ onto S, we have that $\tilde{B}_1 := B_1 \circ R$, $\tilde{B}_2 := B_2 \circ R$ satisfy the uncertainty relation

$$Var(\tilde{B}_1, \tilde{\alpha})\ Var(\tilde{B}_2, \tilde{\alpha}) \geq h \tag{6}$$

for all $\tilde{\alpha} \in \tilde{S}$ at which the two variances exist, with the same h. In other words, the uncertainty relations remain invariant when we move from the observables on S to their representatives on $\tilde{S}$.

3. If two observables B_1, B_2 on S are comeasurable [see (vi) of previous section], then also their representatives $\tilde{B}_1 := B_1 \circ R$ and $\tilde{B}_2 := B_2 \circ R$ on $\tilde{S}$ are comeasurable. Indeed, if B is a joint observable of B_1 and B_2, then $\tilde{B} := B \circ R$ is clearly a joint observable of $\tilde{B}_1$ and $\tilde{B}_2$. Notice, however, that this argument cannot be reversed: as we shall see in the next section, it may happen that two observables on S which are not comeasurable have comeasurable representatives on $\tilde{S}$. This phenomenon shows that the extension procedure pushes toward "less quantal" or "more classical" models. Indeed we shall see that any quantumlike set S of states admits an extension in which the set $\tilde{S}$ is of classical nature, and all observables on it are comeasurable.

4. CLASSICAL EXTENSION OF QUANTUM THEORY

Let the $\tilde{S}$-based model be an extension of the S-based model in the sense said in the previous section. We say that the $\tilde{S}$-based model is the canonical classical extension of the S-based model when $\tilde{S}$ consists of all the probability measures on the set ∂S of the pure states (i.e., the extremal elements) of S; in other words, when there exists an affine surjective map

$$R_M\colon \quad M_1^+(\partial S) \to S \tag{7}$$

The label M refers to Misra (1974), who studied such a map in the particular case in which S is the set S_Q of all density operators on a Hilbert space. The above notion of classical extension was already considered in a more specific context by Holevo (1982).

We list below a few relevant facts pertaining to the notion of canonical classical extension (Beltrametti and Bugajski, 1994).

(i) The S-based model admits the canonical classical extension if S is the set of all countable convex combinations of pure states, in which case the affine surjective map R_M: $M_1^+(\partial S) \to S$ becomes a one-to-one correspondence when restricted to the extremal elements. The image of $\nu \in M_1^+(\partial S)$ under R_M can be written as

$$R_M(\nu) = \int_{\partial S} \alpha \, d\nu(\alpha) \tag{8}$$

where the integral on the right-hand side has to be understood, when ν is concentrated at an infinite number of points of ∂S, as the weak integral, i.e., as the function which attaches to any effect a on S the result of integrating its restriction to ∂S with respect to ν. While the map R_M is one-to-one when restricted to the pure elements, it is in general many-to-one when applied to nonpure elements: this is the case whenever the decomposition of a nonpure state of S into pure states is nonunique. It is worth stressing that the construction of the canonical classical extension is uniquely defined by the fundamental convex structure of S and does not depend on any particular realization of S.

(ii) If B: $S \to M_1^+(\Xi)$ is an observable of the original S-based model, then its representative $\tilde{B} := B \circ R_M$: $M_1^+(\partial S) \to M_1^+(\Xi)$ in the canonical classical extension is regular. Comparing this property with the one quoted at the end of Section 2, we come to the following conclusion: The representatives, in the canonical classical extension, of any two observables of the original S-based model are comeasurable.

(iii) If there exists a pure state of S having dispersion on the observable B: $S \to M_1^+(\Xi)$ of the S-based model, then the representative of B in the canonical classical extension cannot be sharp. This shows that sharpness is not preserved by the canonical classical extension. The representatives of the S-based model acquire the classical feature expressed by the item (ii) above at the price of losing the possibility of being sharp.

On the basis of what is said above, we can now focus attention on the case of quantum mechanics, namely on the case in which S is the set S_Q of density operators on a Hilbert space $\mathcal{H}$ and its boundary ∂S is the set of the one-dimensional projectors. The relevant fact is that S_Q meets the conditions quoted in item (i) above, which means that quantum mechanics does admit the canonical classical extension. In other words, there exists an affine surjective map

$$R_{\mathrm{M}}\colon \quad M_1^+(\partial S_{\mathrm{Q}}) \to S_{\mathrm{Q}} \tag{9}$$

which carries the canonical classical extension of quantum mechanics.

It is worth recalling that the decomposition of a density operator into a convex combination of one-dimensional projectors (the pure states of S_{Q}) is never unique: there are infinitely many distinct convex combinations of pure states that give rise to the same density operator. Therefore, the counterimage of a density operator under R_{M} consists of all the probability measures on the pure states corresponding to that density operator.

Going through the results quoted above (see also Section 3), we can now list a number of facts about observables. For short we call Q-observables the ones on S_{Q}; as seen in Section 2, they include the usual observables described by self-adjoint operators on $\mathcal{H}$, as well as the unsharp ones associated with POV-measures. We call C-representatives their counterparts in the canonical classical extension, namely their representatives on $M_1^+(\partial S_{\mathrm{Q}})$.

1. The statistical distribution of results of a Q-observable is the same as the statistical distribution of results of its C-representative: in short, $BS_{\mathrm{Q}} = \tilde{B}M_1^+(\partial S_{\mathrm{Q}})$ for any Q-observable B. In particular, a Q-observable and its C-representative have the same spectrum and the same eigenvalues; moreover, expectation values and variances are unchanged going from Q-observables to their C-representatives.

2. Whenever two Q-observables obey an uncertainty relation, so do their C-representatives (with the same uncertainty limit).

3. The C-representatives of the Q-observables are regular but not sharp.

4. Any two Q-observables have comeasurable C-representatives.

Summing up, we see that when we represent quantum observables within the canonical classical extension of quantum mechanics, all the most typical quantum features are preserved; nevertheless, some traditional tenets like the claim that uncertainty relations and noncomeasurability always go together have to be dismissed. The representatives of quantum observables in the classical extension are something different from the observables occurring in usual classical statistical mechanics, where only sharp observables are used: in our representation only nonsharp observables occur as images of the quantum observables.

The canonical classical extension of quantum theory calls into play a richer set of states—not just S_{Q}, but all the probability measures on ∂S_{Q}—and a richer set of observables, only some of them being C-representatives of Q-observables. In this sense we might pictorially say that the canonical classical extension is somewhat like a *hidden-observable* generalization of quantum theory. The familiar Q-observables do not separate different convex combinations of pure states that correspond to the same density operator; in other words, there are distinct preparation procedures of statistical ensembles that are not distinguished by the Q-observables. On the contrary, the new observables entering the canonical classical extension do that separation. Thus, what is physically interesting is whether there might exist phenomena that are able to separate those different statistical ensembles which the usual quantum observables are unable to separate; in other words, whether there might be phenomena that make use of the richer set of observables involved in the canonical classical extension. As pointed out years ago by Mielnik (1974), one might conjecture about nonlinear phenomena as possible candidates: indeed, it has been shown (Bugajski, 1992) that the observables described by nonlinear operators find an appropriate representation as observables related to the canonical classical extension.

ACKNOWLEDGMENTS

One of the authors (S.B.) acknowledges the support provided by the Science Research Committee (KBN, Warsaw) under grant 2/2420/92/03.

REFERENCES

Ali, S. T., and Prugovecki, E. (1977). *Journal of Mathematical Physics*, **18**, 219–228.

Amann, A. (1993). The quantum mechanical measurement process in the thermodinamical formalism, in *Symposium on the Foundations of Modern Physics, 1993*, P. Busch, P. Lahti, and P. Mittelstaedt, eds., World Scientific, Singapore.

Beltrametti, E. G., and Bugajski, S. (1993). *International Journal of Theoretical Physics*, **32**, 2235–2244.

Beltrametti, E. G., and Bugajski, S. (1994). A classical extension of quantum mechanics, to be published.

Bugajski, S. (1992). *International Journal of Theoretical Physics*, **30**, 961–971.

Bugajski, S. (1993). *International Journal of Theoretical Physics*, **32**, 969–977.

Cassinelli, G., and Lahti, P. J. (1993). *Journal of Mathematical Physics*, **34**, 5468–5475.

Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press, New York.

Ghirardi, G. C., Rimini, A., and Weber, T. (1976). *Nuovo Cimento*, **36**, 97–118.

Holevo, A. S. (1982). *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam.

Mielnik, B. (1974). *Communication in Mathematical Physics*, **37**, 221–256.

Misra, B. (1974). In *Physical Reality and Mathematical Description*, C. P. Enz and J. Mehra, eds., Reidel, Dordrecht, pp. 455–476.

Singer, M., and Stulpe, W. (1992). *Journal of Mathematical Physics*, **33**, 131–142.